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ONE-DIMENSIONAL ELECTROHYDRODYNAMIC FLOWS WITH SHOCK WAVES

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Stationary one-dimensional electrohydrodynamic flows with shock waves in channels are analyzed for the case of an arbitrary interaction parameter. The criteria of existence of electrohydrodynamic flows with shock waves in which the electric field component normal to the wave front is continuous and that of existence of flows with discontinuities of the electric field in the shock wave are derived. A diagram is constructed in the velocity-electric field coordinate plane which makes it possible to determine the feasibility of obtaining one kind of flow or another by the velocity and the electric field intensity upstream of the shock wave front. An electrohydrodynamic shock adiabatic curve is constructed for a perfect gas.

1. Let us consider the one-dimensional stationary flow of a compressible nonheat-conducting gas with a bulk charge in an electric field. We assume that the velocity of gas and the electric field are in the direction of the x-axis and that all parameters depend only on the x-coordinate. In this case the system of electrohydrodynamic equations consists of the following integrals [1]:

$$\rho u = m = \text{const}, \quad mu + p - \frac{E^2}{8\pi} = \Pi = \text{const}$$
$$m\left(c_pT + {}^1/{}_2u^2\right) + j_0\varphi = \varepsilon = \text{const}, \quad j_0 = q(u + bE) = \text{const} \tag{1.1}$$

where ρ is the medium density, u is the velocity $(u > 0)$, p is the pressure, q is the density of the electric bulk charge $(q > 0)$, E is the electric field intensity φ is the electric potential, j_0 is the electric current density, T is the temperature, and c_p is the specific heat at constant pressure.

In our investigation of one-dimensional electrohydrodynamic flows with shock waves we shall use conventional gasdynamic methods by specifying the shock wave position at $x = \xi$ and, depending on the wave position, determine parameters at the channel outlet. In a supersonic stream the gasdynamic parameters can be specified at the channel inlet, while for the electric field a boundary value problem is to be formulated ($\varphi = 0$ for $x = 0$, and $\varphi = \varphi^*$ for $x = L$, where L is the channel length, and j_0 is specified). Let us use Eq. (1.1) for deriving relationships at the shock wave. We have

$$\rho_1u_1 = \rho_2u_2 = m, \quad mu_1 + p_1 - \frac{E_1^2}{8\pi} = mu_2 + p_2 - \frac{E_2^2}{8\pi} = \Pi$$
$$m\left(c_pT_1 + {}^1/{}_2u_1^2\right) = m\left(c_pT_2 + {}^1/{}_2u_2^2\right) = \varepsilon_g, \quad j_0 = \text{const} \tag{1.2}$$
$$p_2 = \rho_2RT_2$$

where subscripts 1 and 2 denote parameters ahead and behind the discontinuity surface and ε_g is the gasdynamic energy. The last of Eqs. (1.2) is the equation of state. The condition of continuity of electric potential $\varphi_1 = \varphi_2$ was used in the derivation of

Eqs. (1.2) and the parameters upstream of the shock wave were assumed to be known. For obtaining an unambiguous solution of Eqs. (1.2) it is necessary to have one more relationship, since the number of unknowns appearing in (1.2) exceeds the number of equations by one.

In a number of cases the condition of continuity of the electric field component $E_1 = E_2$ normal for $x = \xi$ to the wave front may be used as the required additional relationship. There are, however, cases in which the use of this condition results in a contradiction. In fact, when $E < 0$, $j > 0$, $q > 0$ and $u > 0$, the velocity in the shock wave, by Ohm's law $j = q(u + bE)$, may change to such an extent that in a continuous electric field $u + bE$ changes its sign. This violates the condition of continuity of the electric current density component normal to the wave front. It was shown in [2] that in this case it is necessary to stipulate the fulfillment of condition $u_2 = -bE_2$ downstream of the shock wave, which implies a discontinuity of the electric field normal component and the presence of a surface charge at the shock wave front.

When $j_0 < 0$ and $q > 0$, it is always possible to use the condition of continuity of the electric field. In fact, if the inequality $u + bE < 0$ is valid upstream of the shock wave, it becomes even more stringent downstream of it, since the stream velocity decreases during its passage through the shock wave. It is evident that for $E > 0$ it is always necessary to use the condition of the electric field continuity.

2. Let us analyze the equations of system (1.2) which define the state at the shock wave. Eliminating in these p, ρ and T, we obtain [1]

$$u^2 - \frac{2\gamma\,(\Pi + E^2/8\pi)}{m\,(\gamma + 1)}\,u + \frac{2\,(\gamma - 1)}{m\,(\gamma + 1)}\,\varepsilon_g = 0 \qquad (2.1)$$

Note that Eq. (2.1) is independent of the mobility coefficient b. For fixed flow rate m and total momentum Π Eq. (2.1) yields in the uE-plane (Fig. 1) a set of curves which depend on a single parameter. Lines $M = 1$ and $M = \infty$ (M is the Mach number) are also shown in Fig. 1. The parabola $M = 1$ at each of its intersection points with

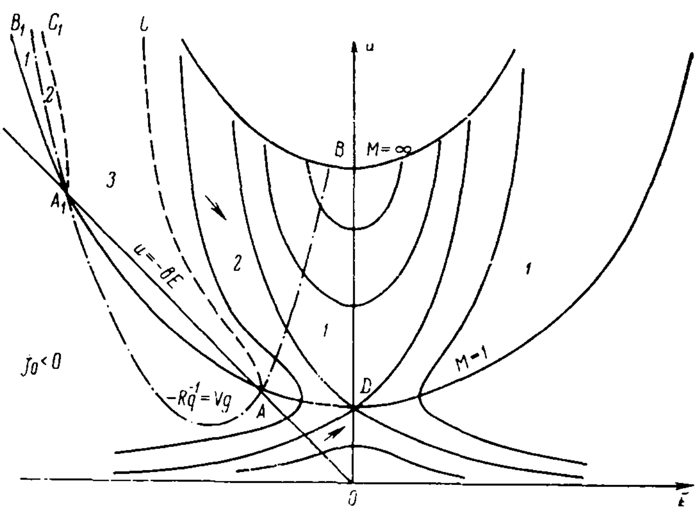

Fig. 1

curves of the set defined by (2.1) has a vertical tangent, with the exception of point D (a singular point of the saddle type) through which pass two singular curves of set (2.1).

Let us define the physical meaning of curves (2.1). We select one of the curves of the set (2.1) by fixing parameter ε_g. Let a point of this curve relate to the state upstream of the shock wave. All of the remaining points of this curve relate to all possible states downstream of the wave. The unambiguous selection of the state downstream of the shock wave requires that one of the two supplementary conditions, defined previously, be satisfied. To determine which of these conditions is to be used we eliminate from Eq. (2.1) parameter E by using the relationship $u = -bE$. We obtain

$$SV(R_q^2V^2 - 1) - P_g(V) = 0 \qquad (2.2)$$

$$S = \frac{E_1^2}{8\pi\rho_1 u_1^2}, \quad R_q = \frac{u_1}{b|E_1|} > 1, \quad V = \frac{u_2}{u_1} = \frac{\rho_1}{\rho_2}$$

$$P_g(V) = \frac{\gamma+1}{2\gamma}V^2 - \left(1 + \frac{1}{\gamma M_1^2}\right)V + \frac{1}{\gamma}\left(\frac{\gamma-1}{2} + \frac{1}{M_1^2}\right)$$

With known solution of the cubic equation (2.2) we can determine all parameters downstream of the shock wave. In the particular case of $S \to 0$ we obtain the condition $P_g(V) = 0$ which corresponds to gasdynamic discontinuities. The polynomial $P_g(V)$ has two roots $V = 1$ and $V = V_g < 1$.

Let us qualitatively analyze Eqs. (2.2) and consider the behavior of function $y = y_1 + y_2$, where along segment $0 \leqslant V \leqslant 1$ $y_1 = SV(R_q^2V^2 - 1)$ and $y_2 = P_g(V)$.

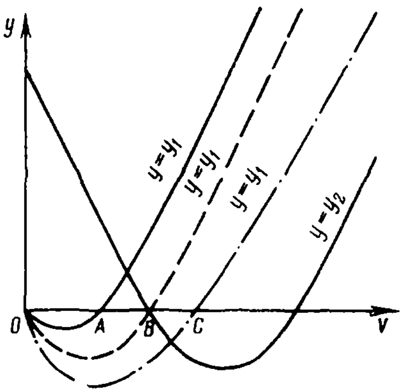

Fig. 2

Along the segment $[0, 1]$ function $y(V)$ changes its sign ($y(0) < 0$ and $y(1) > 0$, since $R_q > 1$). Hence along that segment the cubic polynomial (2.2) has always at least one real root. If $S \geqslant 1 + 1/\gamma M_1^2$ $(\Pi < 0)$, then, by the Descartes rule of signs the polynomial (2.2) has only one positive real root along the segment $[0, 1]$. The case of $S < 1 + 1/\gamma M_1^2$ $(\Pi > 0)$ is examined in detail below.

Curves of y_2 and y_1 are shown in Fig. 2 for various values of parameter R_q (the solid line relates to $R_q^{-1} < V_g$, the dash line to $R_q^{-1} = V_g$, and the dash-dot line to $R_q^{-1} > V_g$).

When $(R_q^{-1} < V_g)$ Eq. (2.2) has only one root along segment AB $(R_q^{-1} < V < V_g)$ and the surface charge density at the shock wave front is negative ($\sigma = E_2 - E_1 < 0$). Thus the use in this case of the condition $u_2 = -bE_2$ downstream of the shock wave results in a physical contradiction, hence it is necessary to use here the condition of continuity of the electric field at the shock wave front.

If $(R_q^{-1} = V_g)$, then $V = V_g = R_q^{-1}$ is the exact solution of Eq. (2.2). This solution corresponds to a flow with a gasdynamic shock wave in which the electric field is continuous and the condition $u_2 = -bE_2$ is satisfied.

Finally, when $(R_q^{-1} > V_g)$, Eq. (2.2) can have along segment BC either one or three real roots.

Let us prove that, depending on parameters S, R_q, γ and M, it can have three roots along segment BC. To do this we chose parameters R_q and M so that the abscissas of minima of curves y_1 and y_2 coincide. Since the equation $y = y_2$ is independent of S, hence by a suitable selection of S we can make curves y_1 and y_2 tangent at the point of minimum, which corresponds to the presence of a multiple root of the cubic equation and, thus, proves the above statement.

Let us represent the obtained results in the uE-plane. The parabola $V_g = -R_q^{-1}$ shown in Fig. 1 by the dash-dot line is defined in variables u, E by the equation

$$u = bE + \frac{2\gamma}{m(\gamma+1)}\left(\Pi + \frac{E^2}{8\pi}\right) \tag{2.3}$$

In the uE-plane the region lying outside parabola (2.3) (region (1) in Fig. 1) corresponds to parameters u and E upstream of the shock wave for which it is necessary to specify the condition of continuity of the electric field $E_1 = E_2$ as the supplementary equation. In this case there is only transition from the supersonic to the subsonic region. The region contained inside the parabola corresponds to such parameters u and E upstream of the shock wave for which it is no longer possible to use the condition of continuity of the electric field in the shock wave. The condition $u_2 = -bE_2$ may be used in this case as the supplementary equation.

To make this clear let us consider in detail the particular case of relative position of curves $M = 1$, $M = \infty$, $V_g = -R_q^{-1}$ and $u = -bE$ shown in Fig. 1, which corresponds to a single root of polynomial (2.2) (each curve of set (2.1) intersects line $u = -bE$ only once). Regions ABC and $A_1B_1C_1$ (regions (2)) are bounded by lines $M = \infty$, $V_g = -R_q^{-1}$, and curves of set (2.1) which pass through points A and A_1 (the dash lines in Fig. 1). In regions (2) transitions with electric field discontinuities from the supersonic region are possible into the subsonic region only. Points lying along segments AO and A_1F of line $u = -bE$ correspond to states downstream of the shock wave, where the flow is subsonic. The direction of variation of stream parameters shown in Fig. 1 by arrows conforms to that in [1]. Region AA_1C_1C (region (3)) corresponds to transition from supersonic states upstream of the shock wave to supersonic states downstream of the latter. Such transitions are nonevolutionary [2].

Besides the configuration of curves $M = 1$, $M = \infty$, $V_g = -R_q^{-1}$ and $u = -bE$, shown in Fig. 1, other relative positions of these curves are possible, which correspond to three roots of the cubic polynomial (2.2) (curves of set (2.1) can intersect the straight line $u = -bE$ three times). Let us prove that in that case there can be only one subsonic root. It follows from (2.1) that for $M < 1$ the derivative u_E' is positive along the lines of this set. Line $u = -bE$ has everywhere a negative slope, hence, if intersection exists in the subsonic region, it must be unique.

It will be seen from Fig. 3 that for $M < 1$ the singular solution intersects line $u = -bE$ for any values of the mobility coefficient b. Let us prove that cases in which one of the roots of Eq. (2.2) is subsonic and the other supersonic are possible. For this it will be sufficient to show that for $M > 1$ the singular curve of Eq. (2.1) intersects line $u = -bE$. Let us take in the supersonic region an arbitrary point P of the singular curve and draw a straight line from that point to the coordinate origin O. This line can be made to coincide with line $u = -bE$ by a suitable choice of the mobility coefficient b (it is determined by the slope of this line). This is always feasible, since the equation of the singular line (2.1) is independent of b. It has been already shown that the singular solution intersects line $u = -bE$ in the subsonic region only once (point R) and, by definition, it must intersect that line in the supersonic region (point P), hence it must intersect the latter for a third time in the supersonic region (point Q).

3. The above analysis can, also, be applied to the derivation of the shock adiabatic curve in the PV-plane. The equations of conservation of momentum and energy presented in dimensionless form are

$$P = 1 + \gamma M_1^2 (1 - V) + S^* (R_q^2 V^2 - 1) \quad P = p_2/p_1 \tag{3.1}$$

$$P = \left(1 + \frac{\gamma - 1}{2} M_1^2\right) \frac{1}{V} - \frac{\gamma - 1}{2} M_1^2 V, \quad S^* = \frac{E_1^2}{8\pi p_1}$$

Eliminating M_1^2 from Eqs. (3.1), we obtain

$$P = \frac{\gamma + 1 - (\gamma - 1) V - S^* (\gamma - 1)(1 + V)(R_q^2 V^2 - 1)}{(\gamma + 1) V - (\gamma - 1)} \tag{3.2}$$

Equation (3.2) yields in electrohydrodynamics a set of shock adiabatic curves which depend on two parameters S^* and R_q. Investigation of the shock adiabatic curve in electrohydrodynamics on the basis of condition $u_2 = -bE_2$ was performed in [3] with the assumption that $R_q^2 V^2 \ll 1$. This inequality is, however, impossible, since $R_q^2 V^2$ is bounded from below by the inequality

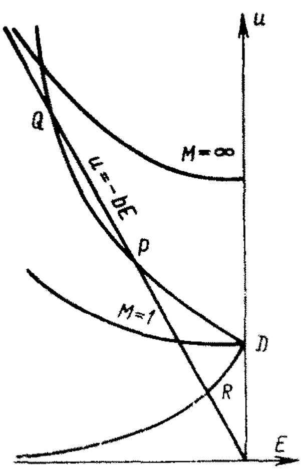

Fig. 3

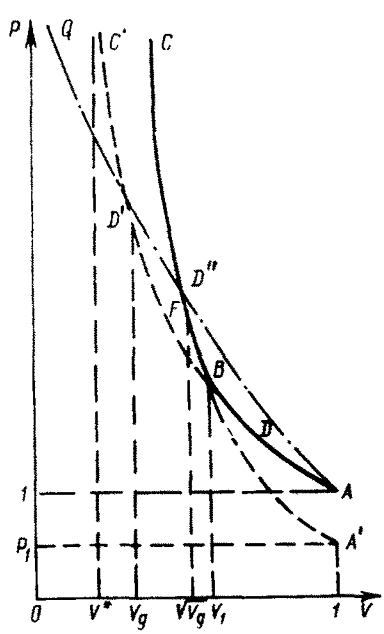

Fig. 4

$$R_q^2 V^2 \geqslant R_q^2 V_q^2 \geqslant R_q^2 (\gamma - 1)^2 / (\gamma + 1)^2$$

and R_q^2 is always greater than unity. This assumption has resulted in wrong conclusions.

Setting in (3.2) $V = 1$, we obtain $P_1 = 1 - S^* (\gamma - 1)(R_q^2 - 1)$ which shows that the shock adiabatic curve (3.2), in this case, lies below the Hugoniot adiabatic curve. The adiabatic curve (3.2) has a vertical asymptote which coincides with that of Hugoniot $V = V^* = (\gamma - 1) / (\gamma + 1)$. The behavior of the adiabatic curve for $V < 1$ depends on parameters S^* and R_q. When $R_q^{-1} < V^*$, the adiabatic curve (3.2) for

$V^* < V < 1$ lies below the Hugoniot curve for any S^*. As was shown above, the use of condition $u_2 = -bE_2$ results in this case in the formation of a negative surface charge at the wave front for all V in the indicated range. This has no physical meaning, hence for $R_q^{-1} < V^*$ it is necessary to use the Hugoniot adiabatic curve (line $ABD'C$ in Fig. 4). When $R_q^{-1} > V^*$ the Hugoniot adiabatic curve intersects the electrohydrodynamic adiabatic curve $A'BD''C$ at point B ($V = V_1 = R_q^{-1}$). For $V < V_1$ the electrohydrodynamic curve lies above the Hugoniot curve (section $BD''C$ in Fig. 4). Thus, for fixed parameters S^* and R_q and arbitrary M_1 the shock adiabatic curve in electrohydrodynamics consists of two segments: one which for $V > V_1$ coincides with the Hugoniot adiabatic curve (segment AB in Fig. 4), the other which for $V < V_1$ is determined by Eq. (3.2) (segment $BD''C$ in Fig. 4).

Let point A relate to the state upstream of the shock wave. We fix M_1 and determine V_g

$$V_g = \frac{2}{\gamma+1}\left(\frac{\gamma-1}{2} + \frac{1}{M_1^2}\right) \tag{3.3}$$

If $V_g \geqslant V_1$, the parameters downstream of the shock wave are determined by conventional gasdynamics formulas (point D in Fig. 4 corresponds to the state downstream of the shock wave). If $V_g < V_1$ (point D' of the Hugoniot adiabatic curve), the state downstream of the shock wave is to be calculated by the adiabatic curve (3.2) and the equation of conservation of energy which is independent of parameters R_q and S^*. By drawing through point A the curve defined by the energy equation (line $AD''D'Q$) we obtain the point of its intersection with the adiabatic curve (3.2) (point D'' in Fig. 4).

Depending on parameters S^*, R_q and M_1, as indicated above, there can be, generally speaking, three such points. In that case point D'' corresponds to the state downstream of the shock wave. For a given M_1 it is possible to plot one more characteristic point on the adiabatic curve, which corresponds to $M_2 = 1$ (point F) and at which $V = \sqrt{V_g}$. From the energy equation we have

$$1 - M_2^2 = \frac{T_1}{T_2}\left[1 + M_1^2\left(\frac{\gamma-1}{2} - \frac{\gamma+1}{2}V^2\right)\right]$$

Setting $M_2 = 1$, we obtain $V = \sqrt{V_g}$. Points lying above point F relate to subsonic states downstream of the shock wave. It will be readily seen that in electrohydrodynamic shock waves the entropy always increases.

Points lying to the right of point A' correspond in electrohydrodynamics to discharge discontinuities, which are accompanied by transition from the supersonic to the supersonic modes. Such transitions are nonevolutionary [3], and in their presence a negative surface charge is generated at the shock wave front. The entropy at such discontinuities decreases, since the adiabatic curve (3.2) in this region lies everywhere below the Poisson adiabatic curve.

Let us determine the point of intersection of the adiabatic curve (3.2) with that of Poisson on $PV^\gamma = 1$ in region $V > 1$. We have

$$y(V) = S^* V^\gamma (R_q^2 V^2 - 1) + V^{\gamma+1} - 1 + (\gamma - 1) V (V^{\gamma-1} - 1) / (\gamma - 1) = 0$$

This formula implies that for $V > 1$ and $\gamma > 1$ function $y(V)$ is always positive and that Eq. (3.4) has no positive roots in the indicated region. Since for $V = 1$ the adiabatic curve (3.2) lies below the Poisson adiabatic curve, it must lie below the latter for all $V > 1$.

4. In conclusion we present a brief summary of results of the above analysis, which may serve as a practical guide for numerical calculations of electrohydrodynamic flows with shock waves.

Let us assume that the shock wave position is specified and that all parameters upstream of the shock are known. These parameters are converted to their dimensionless values S, R_q, M_1 and V_g. If $R_q^{-1} \leqslant V_g$, the electric field at the shock wave is continuous and $V = u_2 / u_1 = V_g$ is determined by formula (3.3). If $R_q^{-1} > V_g$, the electric field at the shock wave is discontinuous and its intensity downstream of the wave is determined by the solution of the cubic equation (2.2).

It was shown that the roots of this equation lie in the interval $V_g < V < R_q^{-1}$. If $R_q^{-1} < \sqrt{V_g}$, then Eq. (2.2) has only one solution which corresponds to the transition from the supersonic to the subsonic mode. When $R_q^{-1} > \sqrt{V_g}$, we can establish the following criterion of existence of a subsonic root of Eq. (2.2). For $V = V_g$ the cubic polynomial (2.2) is negative, and for the existence of a subsonic root it is necessary for this polynomial to be positive, when $V = \sqrt{V_g}$. This leads to the condition that

$$R_q^2 > \frac{P_g(\sqrt{V_g}) + S\sqrt{V_g}}{SV^{1.5}} \tag{4.1}$$

Since $P_g(\sqrt{V_g}) < 0$, condition (4.1) is always satisfied for small interaction parameters S and there is always one subsonic root which is close to the gasdynamic root. The other two roots correspond to $V > 1$ (they relate to discharge discontinuities). If condition (4.1) is not satisfied, all three roots lie in the supersonic region, which evidently implies that for such S, R_q and M_1 stationary flows with shock waves do not occur. In the case in which condition (4.1) is satisfied the cubic equation, in spite of its subsonic root $V = V_1$, can have two more supersonic roots V_2 and V_3. It was shown in [2] that, when all three roots correspond to compression shocks, such shocks cannot exist, since they have no structure. This means that it is not enough to find the subsonic root, and that it is necessary to determine the other two roots in order to check the fulfilment of condition $V_i < 1$ $(i = 2, 3)$.

The calculation of one-dimensional electrohydrodynamic flows with shock waves involves the integration of equations of motion [1]

$$\frac{du}{dx} = \frac{4\pi j_0 E M^2 (u - u_1)}{mu(M^2 - 1)(u + bE)}, \quad \frac{dE}{dx} = \frac{4\pi j_0}{u + bE}, \quad u_1 = (\gamma - 1) bE \tag{4.2}$$

downstream of the shock wave with condition $u = -bE$ satisfied and parameters u' and E' tending to infinity. Expansion of the solution of Eq. (4.2) in the neighborhood of the singularity yields formulas for velocity and the electric field, which can be used in calculations of flows downstream of the shock wave in the vicinity of point $x = \xi$

$$u = u_2 + A\sqrt{\frac{8\pi j_0 (x - \xi)}{A + b}}, \qquad E = -\frac{u_2}{b} + \sqrt{\frac{8\pi j_0 (x - \xi)}{A + b}}$$

$$A = \frac{\gamma E_2 M_2^2}{4\pi m (M_2^2 - 1)}$$

Since for $M_2 < 1$ we always have $A > 0$, hence such expansion is possible.

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ONE-DIMENSIONAL ELECTRO-GASDYNAMIC FLOW WITH SHOCK WAVES AND A SMALL ELECTROHYDRAULIC INTERACTION PARAMETER

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The one-dimensional flow of a unipolarly charged gas between an emitter and a collector is considered for a given discontinuous variation of velocity in the working gap (e. g. in the presence of a gasdynamic shock wave and a small parameter of electrohydraulic interaction). The effects of position and intensity of the velocity discontinuity and of the difference of electrode potentials on the flow properties are determined. It is shown that solutions yielding zero surface charges at the discontinuity can only be obtained in a limited range of variation of determining parameters. Outside that range innumerable solutions yielding nonzero surface charges are possible. A classification of solutions is made on the basis of conditions proposed in [1].

1. Let us consider a one-dimensional flow of unipolarly charged medium in the gap $0 \leqslant X \leqslant L$ between flat electrode grids for the following velocity distribution:

$$\mathbf{V} = (V_* u, 0, 0) \qquad u = \begin{cases} 1, & 0 \leqslant x < \xi \\ r, & \xi < x \leqslant 1 \end{cases}$$

$$x = X/L, \qquad V_*, r = \text{const}, \qquad r \leqslant 1$$

In a stationary one-dimensional motion of gas with a shock wave at point ξ (r is then the ratio of densities at the shock wave) such distribution of velocity obtains in the case of small parameter of electrohydraulic interaction, if the electrical forces do not affect the gasdynamic flow pattern.

The distribution of electric potential φ, electric field $E = E_x(x)$, and of bulk charge density q in the region $0 \leqslant x \leqslant 1$ without allowance for diffusion of charged particles is defined by equations

$$\varphi'' = -q, \quad q = i / (u - \varphi'), \qquad i = const, \qquad E = -\varphi' \tag{1.1}$$